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TRANSIENT TEMPERATURES IN A PLATE FROM A GAUSSIAN DISTRIBUTION OF NORMAL HEAT FLUX AND CURRENT FLOW WITH APPLICATION TO THE FREE ARC DISCHARGE

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Abstract—The current density distribution inside of a metal plate is found for a Gaussian distribution of current density flowing normally into one surface. With a simple approximation to the resulting internal current, the heat equation is solved for the temperature increase due to Joule heating in the plate. A relation for the temperature increase due to a Gaussian distribution of surface heat flux is also derived. Since experimental evidence suggests that the Gaussian distribution is a good approximation for the normal component of current density from an arc discharge onto a plate as anode, the results are applied to the argon arc onto a titanium plate. Evaluation of the parameters for the argon arc using the theoretical model of Schoeck and of Eberhart and Seban and calculations from the solution indicate that Joule heating in a plate from an arc discharge is negligible in comparison with surface heating from the arc discharge.

NOMENCLATURE

B, azimuthal component of magnetic induction;
c^*, specific heat of the plate material;
d^*, plate thickness;
$Ei(z)$, exponential integral, see [1];
I_0^*, maximum total arc current;
$\mathbf{j}_s^*$, current density vector;
j_0^*, current density on plate at center of arc;
j_r, j_z, r and z components of current density;
k^*, thermal conductivity of plate material;
k_g^*, thermal conductivity of gas;
k, Bessel transform variable;
k_b, Boltzmann's constant;
Q_0^*, maximum heat flux through plate surface;
$\mathbf{Q}_s^*$, thermal flux vector;
q, dimensionless heat flux through the plate surface;
r_0^*, radius of Gaussian distribution, $\exp(-r^{*2}/r_0^{*2})$;
s, Laplace transform variable;
$S(t)$, time variation of current density;
t^*, time [s];
T_i^*, initial plate temperature;
T_0^*, reference temperature for Joule heating $(j_0^* d^*)^2/k^*\sigma^*$;
T_s^*, reference temperature for surface heating, $Q_s^* d^*/k^*$;
U_j, dimensionless temperature increment from surface heating, $(T^* - T_i^*)/T_s^*$;
V, dimensionless temperature increment from Joule heating, $(T^* - T_i^*)/T_0^*$;
w, dimensionless radius of Gaussian distribution, r_0^*/d^*;
z^*, co-ordinate normal to plate surface;

β, $= \kappa^*/\kappa_m^*$;
γ, $= (\beta s + k^2)^{\frac{1}{2}}$;
$\varepsilon_1, \varepsilon_2$, constants chosen for approximation of j_r and j_z respectively;
κ^*, thermal diffusivity, k^*/ρ^*c^*;
κ_m^*, magnetic diffusivity, $1/\mu^*\sigma^*$;
μ^*, magnetic permeability of plate material;
ρ^*, density of plate material;
ρ_0, density of gas in arc;
ϕ, work function of the plate material;
$\psi_1(z)$, $= \varepsilon_1(1/3 - z^2)$;
$\psi_2(z)$, $= (2\varepsilon_2/3)(1 - z^2)$.

Superscript and subscripts
$*$, denotes dimensional quantities:
$\tilde{F}_v$, denotes order v Bessel transform of F;
$\bar{F}$, denotes Laplace transform of F.

1. INTRODUCTION

LABORATORY measurements of input current density (see for example Eberhart and Seban [3]) suggest that the normal component of current density into a plate anode may be described fairly closely by a Gaussian distribution. In this work, the current density inside an infinite plate resulting from such a normal surface distribution of current density is obtained. With this current as the internal heat source inside the plate, the heat equation for the rise in temperature in the plate is solved. A formula is also derived for the temperature increase in the plate from a Gaussian distribution of normal heat flux onto one surface when the opposite surface is insulated. These solutions make it possible to describe the temperature distribution in a plate used as an anode for a laboratory high pressure argon arc by following the theoretical model of Schoeck [5] or Eberhart and Seban [3].

2. STATEMENT OF THE PROBLEM

A flat plate of uniform mass density ρ^*, thermal conductivity k^*, specific heat c^*, and initial temperature T_i^* is oriented perpendicularly to the center line of a free, axially symmetric electric discharge.† The plate is bounded by planes at $z^* = 0$ and $z^* = d^*$ (the latter being the arc–plate interface) in a cylindrical coordinate system whose positive z^*-axis coincides with the center line of the arc. The thermal and electrical conductivities of the plate material, k^* and σ^*, respectively, are constant and independent of temperature. Beginning at $t^* = 0$, the arc discharge gives rise to a current density $\boldsymbol{j}_s^*$ and a thermal flux $-\boldsymbol{Q}_s^*$ at the arc–plate interface $z^* = d^*$. The opposite side of the plate, $z^* = 0$, is thermally insulated.

Laboratory measurements of input current density (see e.g. Eberhart and Seban [3]) suggest that the normal component of $\boldsymbol{j}_s^*$ on the surface of a plate electrode may be adequately described by a Gaussian distribution of width r_0^*, or by a superposition of Gaussian functions of different widths. Let I_0^* denote the maximum total arc current. The energy flux into the plate is the sum of contributions $\boldsymbol{q}_j^*$ from several different heating processes. The current density and thermal flux at $z^* = d^*$ may then be conveniently represented as

$$\boldsymbol{j}_s^* = (I_0^*/\pi r_0^*)\, S(t^*) \exp(-r^{*2}/r_0^{*2})\, \hat{\boldsymbol{z}},$$

and

$$\boldsymbol{q}_s^* = \sum q_j^* \hat{\boldsymbol{z}} = \sum Q_j^* q_j(r^*, t^*)\, \hat{\boldsymbol{z}},$$

where Q_j^* is the maximum value of the thermal flux due to the jth process, $\hat{\boldsymbol{z}}$ is the unit vector normal to the plate, and q_j is nondimensional. The temporal modulation of the input current is given by the function S, which is positive (negative) when the plate is an anode (cathode).

Since the thermal rather than the electrical response is of primary interest, it is appropriate to nondimensionalize the independent variables by measuring distance in units of d^*, time in units of d^{*2}/κ^*, where $\kappa^* = k^*/\rho^*c^*$ is the thermal diffusivity, and current density in units of $j_0^* = I_0^*/\pi r_0^{*2}$. For a study of Joule heating in

† Dimensional and nondimensional variables are written with and without asterisks, respectively.

the plate, it is convenient to measure temperature in units of $T_0^* = (j_0^* d^*)^2/k^*\sigma^*$, and electric and magnetic induction fields in units of j_0^*/σ^* and $\mu j_0^* d^*$, respectively, where μ^* is the magnetic permeability of the plate material.

In a free, high-current discharge several complex mechanisms are involved in the transfer of heat energy from the arc to the plate. Detailed discussions of electrode phenomena are found in [2–5], but only a brief description will be included here. The magnitude of net thermal flux q_s^* associated with arc current flow onto a plate anode may be expressed as

$$q_s^* = q_e^* + q_{ra}^* + q_\phi^* + q_v^* + q_c^* - q_{re}^* \quad (1)$$

where q_{re}^* and q_{ra}^* are the radiative fluxes from the plate and arc, respectively. The quantity q_e^* is the rate of energy transfer to a unit area of the plate due to electron conduction and is given by

$$q_e^* = 5j^* k_b T_e^*/2e$$

where T_e^* is the electron temperature and k_b is Boltzmann's constant. The quantity j^* is the normal component of surface current density and e is the electron charge. The energy flux q_v^* in equation (1) represents the heating increment associated with the electron passing through the anode drop V_a, or

$$q_v^* = j^* V_a.$$

Energy is also absorbed at the plate surface by the neutralization of the electrons as they enter the metal lattice, or

$$q_\phi^* = j^*\phi$$

where ϕ is the work function of the metal.

Inside the arc the gas is highly conducting and a jet stream is produced by Lorentz forces directed toward the anode plate. Near the plate in the vicinity of the arc centerline the flow is similar to stagnation point flow, and according to Schoeck [5], the convection heating is described by

$$q_c^* = h^*(T_a^* - T_w^*)$$

where T_a^* is the arc temperature at the edge of the boundary layer and T_w^* is the wall temperature. From Sibulkin [8] the stagnation point heat transfer coefficient h^* is related to the Nusselt number Nu by

$$Nu = h^* r^*/k_g^* = 0{\cdot}763\, r^* Pr^{0\cdot4} (\beta/\nu)^{\frac{1}{2}}$$

where k_g^* is the thermal conductivity of the gas, Pr the Prandtl number, β the velocity gradient, and ν the kinematic viscosity. The gas parameters are evaluated at the film temperature which is half way between the wall and jet temperature at the edge of the boundary layer. Since the jet is produced almost exclusively by Lorentz forces, its characteristic speed as found by Maecker [9] is

$$U_a = (\mu^*/\rho_0)^{\frac{1}{2}} j^* r_0^*$$

where r_0^* is the radius associated with the Gaussian distribution of the normal current density and ρ_0 is the gas density. If $U_a/2r_0^*$ is substituted for the velocity gradient and h^* is evaluated, then the convective heat transfer to the plate is

$$q_c^* = k_g^*(0{\cdot}763) Pr^{0\cdot4} (\mu^*/4\nu^2\rho_0)^{\frac{1}{4}} j^{*\frac{1}{2}}.$$

If we assume that j^* varies spatially as the normal surface current density the variation of q^* for the Gaussian distribution

$$j^* = j_0^* \exp(-r^{*2}/r_0^{*2})$$

takes the form

$$q_c^* = Q_c^* \exp(-r^{*2}/2r_0^{*2}).$$

The radius corresponding to the $1/e$ value of the convective heat minimum is 1·414 times the radius of the current density distribution, a not unrealistic assumption (cf Schoeck [5]).

We find it convenient to measure the temperature increase due to the jth heat flux term in units of $T_j^* = Q_j^* d^*/K^*$. When melting or evaporation of the plate material can be neglected, the total temperature change in the plate can be written as

$$T^* - T_i^* = \sum_j T_j^* U_j + T_0^* V \quad (2)$$

where U_j and V are the solutions of the external surface heat flux and internal Joule heat source problems, respectively. The complete boundary value problems for U and V are as follows:

$$\partial^2 U_j/\partial r^2 + \partial U_j/r\partial r + \partial^2 U_j/\partial z^2 - \partial U_j/\partial t = 0, 0 < r < \infty, 0 < z < 1 \quad (3)$$

$$\partial U_j(r,1,t)/\partial z = q_j(r,t), \partial U_j(r,0,t)/\partial z = U_j(r,z,0) = 0 \quad (4)$$

for the contributions from surface heating and

$$\partial^2 V/\partial r^2 + \partial V/r\partial r + \partial^2 V/\partial z^2 - \partial V/\partial t = -\boldsymbol{j}.\boldsymbol{j}, \quad 0 < r < \infty, 0 < z < 1 \quad (5)$$

$$\partial V(r,0,t)/\partial z = \partial V(r,1,t)/\partial z = V(r,z,0) = 0 \quad (6)$$

for internal Joule heating in the plate.

3. SOLUTION OF THE SURFACE HEATING PROBLEM

To solve the differential equation (3) we apply the Laplace transform in time and Bessel transform of zero order in the radial variable r to obtain an ordinary differential equation in z. The Laplace transform of $f(r, z, t)$ is denoted by $\bar{f}(r, z, s)$ while the Bessel transform of order ν is written as $\tilde{f}_\nu(k, z, t)$; i.e.

$$\bar{\tilde{f}}_\nu(k,z,s) \equiv \int_0^t \mathrm{d}t \exp(-st) \int_0^\infty \mathrm{d}r\, r J_\nu(kr) f(r,z,t).$$

With the introduction of this notation the transformed differential equation becomes

$$\partial^2 \bar{\tilde{U}}_j/\partial z^2 - (k^2 + s)\bar{\tilde{U}}_j = 0 \quad (7)$$

and the solution satisfying the appropriate transformed boundary conditions is seen to be

$$\bar{\tilde{U}}_j = \bar{\tilde{q}}_j(k,s)\cosh\sqrt{(k^2+s)}z/\sqrt{(k^2+s)} \times \sinh\sqrt{(k^2+s)}.$$

Performing the inversion with respect to t and expressing the inverse Bessel transform as an integral yield

$$U_j(r,z,t) = \int_0^t \mathrm{d}u \int_0^\infty \mathrm{d}k\, k J_0(kr)\exp(-k^2 t) \times q_j(k, t-u)\,\theta_4(z/2, \exp(-\pi^2 u))$$

where the Theta function follows the definition of Abramovitz and Stegun [1] or

$$\theta_4(z/2, \exp(-\pi^2 u)) \equiv 1 + 2\sum_{n=1}^{\infty}(-1)^n \exp(-\pi^2 n^2 u)\cos(\pi n z).$$

This result may be expressed in more usable form by means of the Macauley–Owen theorem relating two functions f_1 and f_2 to their Bessel transforms, namely

$$\int_0^\infty \mathrm{d}k\, k f_1(k) f_2(k) = \int_0^\infty \mathrm{d}r_0 r_0 f_1(r_0) f_2(r_0).$$

With the aid of the integral formula from Watson ([6], p. 395)

$$\int_0^\infty \mathrm{d}k\, k \exp(-k^2 u) J_0(kr) J_0(kr_0) = \exp(-(r^2 + r_0^2)/4u) I_0(rr_0/2u)/2u$$

we finally obtain

$$U_j(r,z,t) = \int_0^t \frac{\mathrm{d}u}{2u}\int_0^\infty \mathrm{d}r_0 r_0 q_j(r_0, t-u) \times \exp[-(r^2+r_0^2)/4u] I_0(rr_0/2u) \times \theta_4(z/2, \exp(-\pi^2 u)).$$

When q_j is given by a step function in time and a Gaussian distribution in the radial direction $\exp(-r^2/w^2)$, the integration with respect to r_0 can be performed using the integral formula in Magnus, Oberhettinger and Soni ([7], p. 93). The solution of the jth term of the internal heating problem takes the form

$$U_j(r,z,t) = w^2 \int_0^t \mathrm{d}u \exp(-r^2/(w^2+4u)) \times \theta_4(z/2, \exp(-\pi^2 u))/(w^2+4u). \quad (8)$$

4. DERIVATION OF THE CURRENT DENSITY

It is advantageous first to derive the azimuth component of the magnetic field, $B(r, z, t)$, induced by the current flow in the plate. Then, if displacement currents are neglected, the current density in the plate can be calculated from

$$j_r = -\frac{\partial B}{\partial z}, \qquad j_z = \frac{1}{r}\frac{\partial}{\partial r}(rB). \tag{9}$$

Maxwell's equations and Ohm's Law lead to an equation for B in which the only parameter is the ratio $\beta = \kappa^*/\kappa_m^*$, where $\kappa_m^* = 1/\mu^*\sigma^*$ is the magnetic diffusivity:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rB)\right) + \frac{\partial^2 B}{\partial z^2} - \beta\frac{\partial B}{\partial t} = 0,$$

$$0 < r < \infty, 0 < z < 1; \tag{10}$$

$$\frac{1}{r}\frac{\partial(rB)}{\partial r} = \begin{cases} 0 \text{ at } z = 0 \\ S(t)\exp(-r^2/w^2) \text{ at } z = 1 \end{cases} \tag{11}$$

and

$$B(r, z, 0) = 0.$$

The boundary condition at $z = 1$ is the non-dimensional equivalent of (1).

To solve (10), we apply the Laplace transformation and write

$$\bar{B}(r, z, s) = \int_0^\infty dk k J_1(kr)\,\bar{\tilde{B}}_1(k, z, s). \tag{12}$$

The double transform $\bar{\tilde{B}}_1$ will then satisfy the same differential equation as (7). To find the equivalent boundary condition for $\bar{\tilde{B}}_1$ as we note that

$$\exp(-r^2/w^2) = (w^2/2)\int_0^\infty dk k J_0(kr) \times \exp(-w^2k^2/4) \tag{13}$$

from Watson ([6], p. 393). Substitution of (12) and (13) and combining the integrands yield for the boundary condition

$$\bar{\tilde{B}}_1(k, 0, s) = \overline{S(s)}w^2\exp(-k^2w^2/4)/2k.$$

The solution of (7) which satisfies this boundary condition is

$$\bar{\tilde{B}}_1(k, z, s) = \overline{S(s)}(w^2/2k)\exp(-w^2k^2/4) \times [\sinh(\gamma z)/\sinh(\gamma)], \tag{14}$$

where

$$\gamma = (\beta s + k^2)^{\frac{1}{2}}. \tag{15}$$

The parameter β is less than 10^{-2} even for most metals, including good electrical conductors such as copper and aluminium. Throughout the sequel, β is limited to values sufficiently small that only the thermal response for large t/β is of importance. We first find the inverse Laplace transform of $\tilde{B}(k, z, s)$ by evaluating the residues at

$$\gamma^2 = \beta s + k^2 = -n^2\pi^2.$$

Thus we obtain

$$B(r, z, t) = w^2\sum_1^\infty(-1)^{n-1}\pi n\sin(n\pi z)\int_0^\infty dk\,J_1(kr) \times \exp(-w^2k^2/4)\,F_n(k, t; \beta),$$

where

$$F_n(k, t; \beta) = \int_0^{t/\beta} du\,S(t - \beta u) \times \exp[-(k^2 + n^2\pi^2)u]. \tag{16}$$

When the current modulation $S(t)$ is a slowly varying function over time intervals of width β, then

$$F_n(k, t; \beta) \simeq S(t)(k^2 + n^2\pi^2)^{-1} + 0(\beta). \tag{17}$$

With (17) substituted into the integrand of (16), the series can be summed to yield

$$B(r, z, t) \doteq S(t)\frac{w^2}{2}\int_0^\infty dk\,J_1(kr) \times \exp(-w^2k^2/4)\left[\frac{\sinh(kz)}{\sinh(k)}\right]. \tag{18}$$

When this expression is substituted into (6), the result is simply the quasi-stationary current density. From the physical point of view, small values of β imply that plate currents can adjust to changes in input current much more rapidly than the temperature field can accommodate to corresponding changes in Joule heating.

When most of the input current flows through an area whose radius is larger than the thickness of the plate, i.e. when w is large, a simple approximation can be developed for the quadrature in (11). The source term $\boldsymbol{j}\,.\boldsymbol{j}$ is then such that the thermal problem stated in (5) has an analytic solution.

5. SOLUTION OF THE JOULE HEATING PROBLEM

For large w, the exponential in the integrand of (11) falls very rapidly to small values as k increases from zero. Since z does not exceed unity, the bracketed factor is a relatively slowly varying function of k about $k = 0$, and this suggests an expansion in a Maclaurin series:

$$\sinh(kz)/\sinh(k) = z - (k^2/6)z(1 - z^2) + (k^4 z/12)(7/30 - z^2/3 + z^4/10) + 0(k^6). \qquad (19)$$

With $P = w^2/4$, the resulting integrals in (18) take a form which can be integrated by using equation (2), p. 393 and p. 100 of Watson [6]. We obtain

$$\int_0^\infty dk\, J_1(kr)\exp(-Pk^2)k^{2n} = \frac{(-1)^n}{r}\frac{\partial^n}{\partial P^n}[1 - \exp(-r^2/4P)]. \qquad (20)$$

Using (19) and (20) in (18) leads ultimately to the following expansions for the current density components:

$$j_r = -S(t)\{(w^2/2r)[1 - \exp(-r^2/w^2)] - (r/w^2)(1/3 - z^2)\exp(-r^2/w^2) + 0(r/w^4)\}$$
$$j_z = S(t)\{z\exp(-r^2/w^2)[1 - (2/3w^2) \times (1 - z^2)(1 - r^2/w^2) + 0(1/w^4)\}. \qquad (21)$$

Only the first terms of (21) need be retained when w is large compared with unity. When w is not large a useful approximation to the current density can be constructed from (21) by multiplying the second terms in each component by suitably chosen constants ε_1 and ε_2. Thus, (for $t > 0$) we write for a step function increase in current

$$j_r = -(w^2/2r)[1 - \exp(-r^2/w^2)] + \varepsilon_1(r/w^2)(1/3 - z^2)\exp(-r^2/w^2)$$

and (22)

$$j_z = z\exp(-r^2/w^2)[1 - \varepsilon_2(2/3w^2) \times (1 - z^2)(1 - r^2/w^2)].$$

There is no unique procedure for assigning values to ε_1 and ε_2. In the present work, the constants are chosen to give good approximations to the exact components in the neighborhood of the axis of symmetry where most of the resistive heating occurs. Thus, the parameter ε_1 is such that the approximate value of j_r from (22) is equal to the exact value at $r = \frac{1}{2}$ and $z = 0$. Similarly, the value of ε_2 is chosen to be such that the approximation to j_z is equal to the exact value computed from (18) and (19) at $r = 0$ for

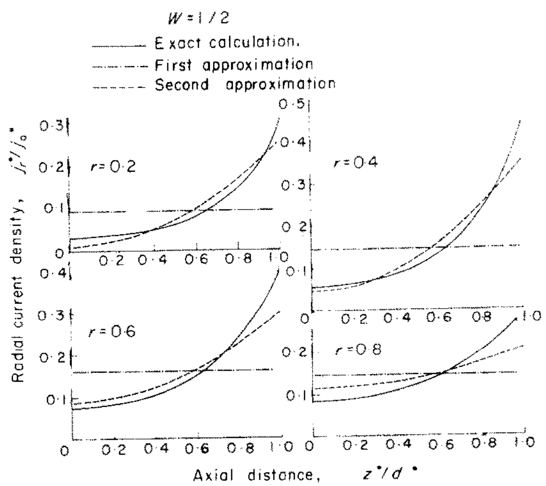

FIG. 1. Variation of radial current density with axial distance at several radial stations for the case $w = \frac{1}{2}$.

the midpoint $z = \frac{1}{2}$. When chosen in this fashion, the constants depend upon the parameter w. By way of example, for the case $w = \frac{1}{2}$, the correspondences are achieved with $\varepsilon_1 = 0{\cdot}370$ and $\varepsilon_2 = 0{\cdot}293$.

In Figs. 1 and 2, the first approximation to j_r and j_z ($\varepsilon_1 = \varepsilon_2 = 0$) and the second approximation from (22) are compared with exact calculations from (18) and (9) for the case $w = \frac{1}{2}$.

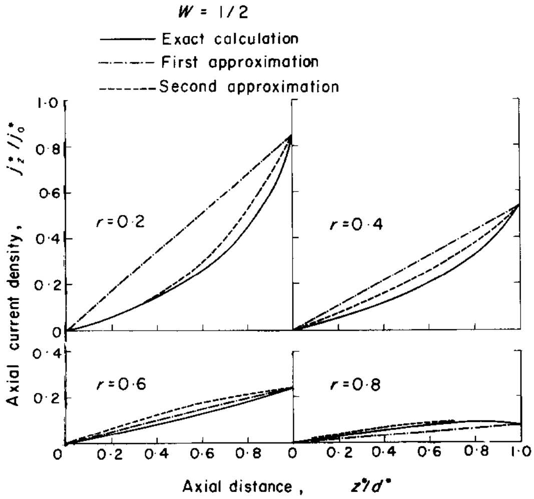

FIG. 2. Variation of axial current density with axial distance at several radial stations for the case $w = \frac{1}{2}$.

Similar calculations are shown in Fig. 3 for $r = 0{\cdot}4$ for the case $w = 1$. More generally, Table 1 displays appropriate values of ε_1 and ε_2 corresponding to several different values of w.

To solve (5) we need to construct the scalar product $\boldsymbol{j}\,.\boldsymbol{j}$ from (22). This product is proportional to the rate of ohmic heating and for convenience will be denoted by $H(z, r)$:

$$H(z,r) = \boldsymbol{j}\,.\boldsymbol{j} = \frac{w^4}{4r^2}[1 - 2\exp(-r^2/w^2) + \exp(-2r^2/w^2)] + z^2\exp(-2r/w^2) - \psi_1\exp(-r^2/w^2) + [\psi_1 w^2 - 2z^2\psi_2 + z^2\psi_2^2/w^2]\exp(-2r^2/w^2)/w^2 + [\psi_1^2 + 2z^2\psi_2 - 2z^2\psi_2^2/w^2]r^2 \times \exp(-2r^2/w^2)/w^4 + (z^2\psi_2^2/w^2)r^4 \exp(-2r^2/w^2)/w^6 \qquad (23)$$

Table 1. Values of ε_1 and ε_2 corresponding to several values of plate thickness w

w	ε_1	ε_2
0·5	0·370	0·293
1·0	0·583	0·590
2·0	0·825	0·838
3·0	0·910	0·918
4·0	0·947	0·952
5·0	0·965	0·969

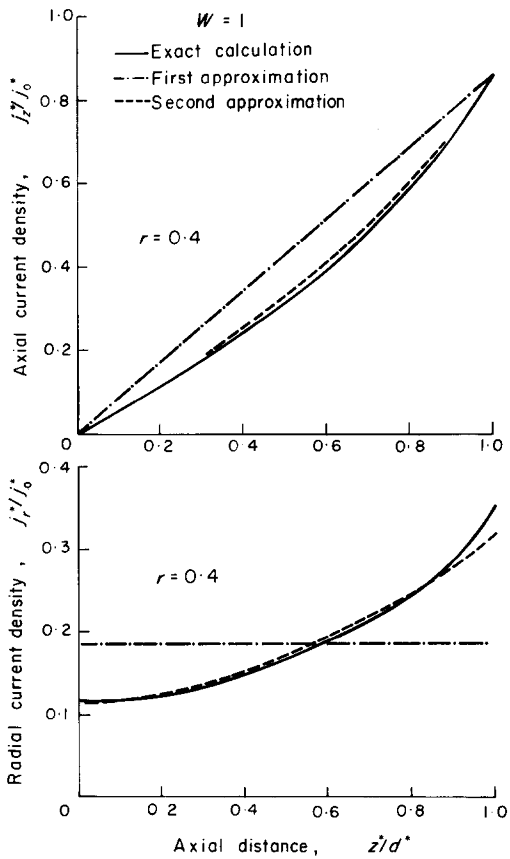

FIG. 3. Variation of axial and radial current density components with axial distance at $r = 0{\cdot}4$ for the case $w = 1$.

where we have introduced the functions

$$\psi_1(z) = \varepsilon_1(1/3 - z^2) \tag{24}$$

$$\psi_2(z) = 2\varepsilon_2(1 - z^2)/3. \tag{25}$$

The term in $1/r^2$ is not in a form convenient for applying the Bessel transform. This can be easily overcome by noting that

$$(1/r^2)[1 - \exp(-\lambda r^2)] = \int_0^\lambda d\eta \exp(-r^2\eta). \tag{26}$$

$H(z, r)$ is then expressed as polynomials in r^2 times negative exponentials in r^2 and its Bessel transform is easily obtained by using the formula on p. 393 of Watson [6] which for our purposes is written

$$\int_0^\infty dr\, J_0(kr) \exp(-Pr^2) r^{2n+1}$$

$$= (-1)^n \frac{\partial^n}{\partial P^n}[(2P)^{-1} \exp(-k^2/4P)]. \tag{27}$$

With the substitution of (23) and (26) into (5), application of the double transform to the resulting differential equation then yields

$$\frac{\partial^2 \tilde{\bar{V}}_0}{\partial z^2} - (s + k^2)\tilde{\bar{V}}_0 = -\tilde{H}_0(z, k)/s \tag{28}$$

subject to

$$\partial \bar{V}_0(s, k, 0)/\partial z = \partial \tilde{\bar{V}}_0(s, k, 1)/\partial z = 0, \tag{29}$$

where

$$\tilde{H}_0(z, k) = (w^4/4)\Big[\int_0^{1/w^2} - \tfrac{1}{2}\int_0^{2/w^2} \exp(-k^2/4\eta)$$

$$\times d\eta/\eta\Big] + (w^2z^2/4)\exp(-w^2k^2/8)$$

$$- (w^2\psi_1/2)\exp(-w^2k^2/4) + (1/4)$$

$$\times (w^2\psi_1 - 2z^2\psi_2 + z^2\psi^2/w^2)\exp(-w^2k^2/8)$$

$$+ (1/8)(\psi_1^2 + 2z^2\psi_2 - 2z^2\psi_2^2/w^2)(1 - w^2k^2/8)$$

$$\times \exp(-w^2k^2/8) + (1/8)z^2\psi_2^2(1 - w^2k^2/4$$

$$+ w^4k^4/128)\exp(-w^2k^2/8)/w^2. \tag{30}$$

The solution of (28) which satisfies the boundary conditions in (29) is

$$\tilde{\bar{V}}_0 = s^{-1}\int_0^z d\eta\, \tilde{H}_0(\eta)[\cosh \mu\eta \cosh \mu$$

$$\times (1 - z)/\mu \sinh \mu] + s^{-1}\int_z^1 d\eta\, \tilde{H}_0(\eta)$$

$$\times [\cosh \mu z \cosh \mu(1 - \eta)/\mu \sinh \mu]$$

where

$$\mu = (s + k^2)^{\frac{1}{2}}. \tag{31}$$

It is convenient first to invert $s\tilde{\bar{V}}_0$ with respect to time and then to perform the Bessel inversion of $\partial\tilde{V}_0/\partial t$. Thus.

$$\partial\tilde{V}_0/\partial t = \int_0^1 d\eta\, \tilde{H}_0(\eta)\exp(-k^2t)\Big[1 + 2\sum_1^\infty$$

$$\times \cos \pi z \cos n\pi\eta \exp(-n^2\pi^2 t)\Big]. \tag{32}$$

The integrals which result from the substitution of (30) into (32) have the form of Fourier coefficients:

$$a_n = \lambda_n \int_0^1 d\eta[w^2\psi_1(\eta)\cos n\pi\eta];\ \lambda_0 = 1, \lambda_n$$

$$= 2 \text{ for } n > 0,$$

and similarly,

$$b_n = \lambda_n \int_0^1 d\eta \cos n\pi\eta[\psi_1^2(\eta)]$$

$$c_n = \lambda_n \int_0^1 d\eta \cos n\pi\eta[2\eta^2\psi_2(\eta)]$$

$$d_n = \lambda_n \int_0^1 d\eta \cos n\pi\eta[(\eta^2/w^2)\psi_2^2(\eta)]$$

$$e_n = \lambda_n \int_0^1 d\eta \cos n\pi\eta\, \eta^2. \tag{33}$$

In terms of these coefficients, (32) takes the form

$$\partial V_0/\partial t = (w^2/4)\Big[\int_0^{1/w^2} - \tfrac{1}{2}\int_0^{2/w^2} \exp(-k^2$$

$$\times (t + 1/4\eta))\, d\eta/\eta\Big] + \sum_{n=0}^\infty \exp(-n^2\pi^2 t)$$

$$\cosh \pi z\{w^2 e_n \exp(-k^2(t + w^2/8))/4$$

$$- (a_n/2)\exp[-k^2(t + w^2/4)]$$

$$+ (a_n - c_n + d_n)\exp(-k^2(t + w^2/8))/4$$

$$+ (b_n + c_n - 2d_n)(1 - w^2k^2/8) \times \exp(-k^2(t + w^2/8))/8 + d_n(1 - w^2k^2/4 + w^4k^4/128)\exp(-k^2(t + w^2/8))/8\}. \quad (34)$$

The Bessel inversion can be accomplished by using (27) with k and r interchanged. After collecting the coefficients of k^2 and k^4, the integrations over k are readily performed; the final result for V can be written as a quadrature:

$$V = \int_0^t du \Big\{ (w^4/2)\Big[\Big(\int_0^{1/w^2} - \tfrac{1}{2}\int_0^{2/w^2}\Big) \times \exp\Big(-\frac{ar^2}{4au+1}\Big)\frac{da}{4au+1}\Big] + \sum_{n=0}^{\infty} \exp(-n^2\pi^2u)\cos n\pi z \times [w^2e_n\exp(-2r^2/\tau_2)/\tau_2 - a_n\exp(-r^2/\tau_1)/\tau_1 + (2a_n + b_n - c_n + d_n) \times \exp(-2r^2/\tau_2)/2\tau_2 - w^2(b_n + c_n) \times \exp(-2r^2/\tau_2)(1 - 2r^2/\tau_2)/2\tau_2^2 + w^4d_n(1 - 4r^2/\tau_2 + 2r^4/\tau_2^2) \times \exp(-2r^2/\tau_2)/2\tau_2^3]\Big\} \quad (35)$$

where

$$\tau_1 = w^2 + 4u$$
$$\tau_2 = w^2 + 8u.$$

The first approximation of V for large w, in which terms in the current density to the order of $1/w^2$ and higher are neglected, is found from equation (35) by setting $\varepsilon_1 = \varepsilon_1 = 0$ or

$$a_n = b_n = c_n = d_n = 0.$$

The integrals inside the curly brackets in (35) can be expressed in terms of exponential integrals by introducing a new variable of integration defined by

$$\zeta = r^2/4u(4au + 1) = r^2/4u - ar^2/(4au + 1). \quad (36)$$

Evaluating the coefficients e_n yields for the first approximation for V

$$V = \int_0^t du \Big\{ \frac{w^4}{16u}\Big[2Ei\Big(\frac{r^2w^2}{4u(4u + w^2)}\Big) - Ei\Big(\frac{r^2w^2}{4u(8u + w^2)}\Big) - Ei(r^2/4u)\Big] \times \exp(-r^2/4u) + w^2\exp(-2r^2/\tau_2) \times [1/3 + 4\sum_{n=1}^{\infty}\cos n\pi z \times \exp(-n^2\pi^2u)/\pi^2n^2]/\tau_2\Big\} + 0(1/w^2) \quad (37)$$

where $Ei(x)$ is the notation for the exponential function used by Abramovitz and Stegun [1]. When r is set equal to zero, (37) is indeterminate but from (35) we see that both integrations of the square bracketed terms can be performed in closed form. The remaining integrals can be expressed in terms of the exponential integrals for which rational approximations exist.

6. RESULTS AND CONCLUSIONS

The results derived in the preceding sections were used to calculate the temporal variation of temperature at $r = z = 0$ due to ohmic heating of the plate. This is the point on the insulated surface where the temperature rise is greatest. In Fig. 4 the nondimensional first and second approximations of the temperature enhancement are shown as a function of time for the step function in the current and for several values of w. It can be seen from the figure that the first approximation to V yields an overestimate of the Joule heating effect over the time interval considered.

It is possible to compare the temperature increase due to Joule heating with that due to external thermal energy transferred from the

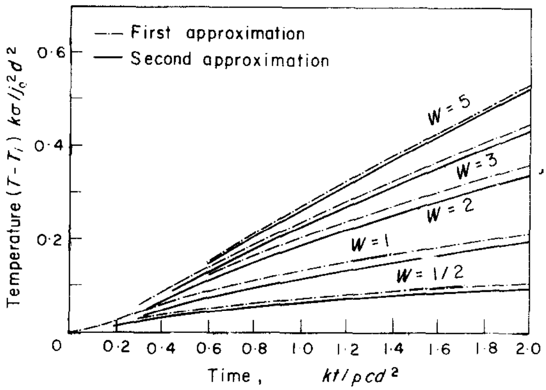

FIG. 4. First and second approximations to the transient temperature at $r = z = 0$ due to Joule heating.

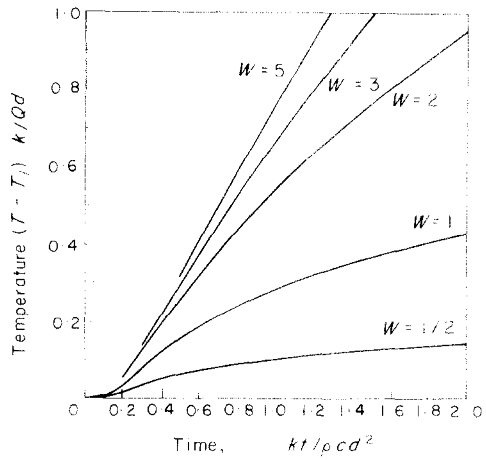

FIG. 5. Transient temperature rise at $r = z = 0$ due to external energy flux from arc discharge.

discharge. All the surface sources of heat described in Section 2 except convection and radiation are approximately proportional to the normal component of current density. Thus, the dimensionless temperature increase from a Gaussian distribution of surface heat influx as given in (8) applies with the same values of w as in the corresponding ohmic internal heating solution (37). Figure 5 shows graphs of the dimensionless temperature increase at the hottest point on the insulated surface as functions of time for several values of w. Comparison of V and U in Figs. 4 and 5 for the same values of w shows that U generally exceeds V at comparable times by a factor between 1·5 and 3. However, the true comparative value of U is obtained by considering the ratio of temperature units for internal Joule heating and for surface heating, i.e. T_0^*/T_s^*. From Section 2, a consideration of the surface heating processes associated with the flow of electrons gives the nondimensional temperature T_s^* as

$$T_s^* = Q_s^* d^*/k^* = j_0^*(5k_b T_e^*/2e + \phi + V_a) \times (d^*/k^*). \tag{38}$$

A measure of the relative magnitude of Joule heating as compared with heating due to the electron flux at the surface is then given by

$$\frac{T_0^*}{T_s^*} = \frac{j_0^* d^*/\sigma^*}{5k_b T_e^*/2e + \phi + V_a}. \tag{39}$$

The assignment of accurate parameter values in (39) is not a straightforward task. The properties of argon arcs have been determined more precisely than those of arcs in other gases, such as nitrogen or air, and for that reason we shall use parameter values appropriate to moderately intense argon arcs in a numerical example. Thus, the electron temperature T_e^* will be taken as 80 per cent of the gas temperature, the latter being in the neighbourhood of 12 000°K (cf. Schoeck [5]) for moderately intense currents. The work function ϕ and the electrical resistivity $1/\sigma^*$ of the plate depend, of course, on the metal of which it is composed. We use here the values $\phi = 4$ V and $1/\sigma^* = 1{\cdot}6 \times 10^{-4}$ ohm-cm, appropriate to titanium. (This choice was motivated by an interest in electrode effects associated with lightning strikes to aircraft with titanium skin.) Then, with an anode drop of the order of 1–2 V (Eberhart and Seban [3]), the denominator of (39) is of the order of 7 V. According to the laboratory study by Eberhart and Seban [3] the centerline current density of a moderately intense arc in argon is a function of the total current I, having a maximum of about 600 A cm^{-2}

and decreasing for higher values of I. If we set j_0^* equal to this value, then for a titanium plate of thickness d^* in the range considered for aircraft wings (0·1–0·2 cm) the numerator of (39) is approximately 0·01–0·02 V. In this particular instance, T_0^*/T_s^* is of the order of 0·001, which indicates that the effect of Joule heating is small relative to the external heating due to electron flow. In making the foregoing estimate, we have not evaluated convective heating which is expected to dominate the total heat input for currents in the kiloampere range.

The foregoing arguments also support the assertion that Joule heating of the anode plate of most laboratory arcs constitutes a relatively small fraction of the total thermal energy input to the plate. However, under some circumstances, the effect of ohmic heating may be of importance and the analysis set forth in Section 5 provides a quantitative description of the effect in such cases. By way of example, in the case of laboratory arcs, the influence of Joule heating will be enhanced if the current density near the center of the arc is increased by "jetting" of metal vapor, or if the plate thickness is increased, or if the plate material is of lower conductivity. Nevertheless, Joule heating of laboratory electrode plates will not become important unless these factors combine to increase T_0^*/T_s^* by two or three orders of magnitude. In different contexts, however, Joule heating can be a significant factor. To cite an example of practical importance, ohmic heating can be an important source of thermal energy when a lightning discharge interacts with materials of high electrical resistivity in certain airplane components.

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TEMPERATURES TRANSITOIRES DANS UNE PLAQUE À PARTIR D'UNE DISTRIBUTION GAUSSIENNE DE FLUX THERMIQUE NORMAL ET DE COURANT AVEC APPLICATION À LA DECHARGE LIBRE D'UN ARC

Résumé—La distribution de densité de courant à l'intérieur d'une plaque métallique est déterminée pour une distribution gaussienne de densité de courant s'écoulant normalement vers une surface. On a résolu à partir d'une approche simple du courant interne résultant, l'équation de chaleur pour l'accroissement de température par effet Joule dans la plaque. On obtient aussi une relation pour l'augmentation de température due à la distribution gaussienne du flux thermique en surface. Puisque l'évidence expérimentale suggère que la distribution de Gauss est une bonne approximation pour la composante normale de la densité de courant de décharge d'un arc à une plaque anode, les résultats sont appliqués à un arc d'argon vers une plaque de titane. L'évaluation des paramètres pour l'arc d'argon utilisant le modèle théorique de Schoeck et d'Eberhart et Seban ainsi que les calculs à partir de la solution montrent que le chauffage par effet Joule dans la plaque est négligeable par rapport à l'échauffement de la surface par la décharge de l'arc.

INSTATIONÄRES TEMPERATURFELD IN EINER PLATTE BEI WÄRMESTROMDICHTE UND ELEKTRISCHER STROMDICHTE NACH EINER GAUSS-VERTEILUNG. ANWENDUNG AUF DIE LICHTBOGENENTLADUNG.

Zusammenfassung—Die Verteilung der elektrischen Stromdichte in einer Metallplatte wird für den Fall bestimmt, dass die normal zu einer Oberfläche aufgeprägte Stromdichte einer Gauss-Verteilung folgt. Mit einer einfachen Approximation für die sich im Innern einstellende Stromdichte wird die "Wärmeleitungsgleichung zur Bestimmung des Temperaturanstiegs infolge Joulescher Aufheizung der Platte gelöst. Ebenso wird eine Beziehung für den Temperaturanstieg infolge eines nach einer Gauss-Verteilung aufgeprägten Oberflächenwärmeflusses abgeleitet.

Da Experimente erkennen lassen, dass die Gauss-Verteilung eine gute Näherung für den Verlauf der Normalkomponente der elektrischen Stromdichte in einer Platte darstellt, wenn diese bei einer Lichtbogenentladung als Anode dient, werden die Ergebnisse auf den Fall eines auf eine Titanplatte treffenden Argonlichtbogens angewendet. Die Parameter für den Argonlichtbogen wurden mit Hilfe des theoretischen Modells von Schoeck bzw. von Eberhart und Seban bestimmt und damit die gefundenen Lösungen ausgewertet. Es zeigt sich, dass die Joule'sche Aufheizung einer Platte infolge einer Lichtbogenentladung gegenüber der vom Lichtbogen an die Oberfläche übertragenen Wärmeenergie vernachlässigt werden kann.

ТЕМПЕРАТУРНОЕ ПОЛЕ В ПЛАСТИНЕ ПРИ ГАУССОВОМ РАСПРЕДЕЛЕНИИ НОРМАЛЬНОГО ТЕПЛОВОГО ПОТОКА И ПЛОТНОСТИ ТОКА В СВОБОДНОМ ДУГОВОМ РАЗРЯДЕ

Аннотация—При гауссовом распределении нормальной компоненты плотности тока на одной из поверхностей металлической пластины получено распределение плотности тока внутри неё.

Используя простую аппроксимацию для выражения относительного результирующего тока, решается уравнение теплопроводности, позволяющее определить увеличение температуры в пластине за счет нагрева джоулевым теплом. Получено соотношение для расчета повышения температуры вследствие распределения теплового потока на поверхности по гауссовому закону. Поскольку эксперимент показал, что гауссово распределение является хорошей аппроксимацией зависимости нормальной компоненты плотности тока от дугового разряда на пластине, используемой в качестве анода, результаты использованы для аргоновой дуги на титановой пластине. Оценка параметров аргоновой дуги по теоретической модели Шоека-Эберхарта и Себана, а также полученные на основе решения расчеты показывают, что нагрев пластины джоулевым теплом пренебрежимо мал по сравнению с нагревом поверхности от дугового разряда.